Chap 14.9 Taylor's Formula for Two Variables

1 (Taylor's formula for $f(x,y)$ at the point (a,b))

Suppose $f(x,y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at a point (a,b). Then, through R,

$$f(a+h,b+k) = f(a,b) + (hf_x + kf_y)\big|_{(a,b)} + \frac{1}{2}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)}$$
$$+ \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)} +$$
$$\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch,\, b+ck)}.$$

2. The error estimates for linear approximations.

$$E(x,y) := f(x_0,y_0) - L(x_0,y_0) \quad (L(x_0,y_0) = (x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0))$$

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2.$$

Exercise

Find the quadratic approximations of $f(x,y) = \sin(x^2+y^2)$ near origin.

Solution: $f_x = 2x\cos(x^2+y^2)$ $\quad f_y = 2y\cos(x^2+y^2)$

$$f_{xx} = 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)$$

$$f_{xy} = -4xy\sin(x^2+y^2)$$

$$f_{yy} = 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2)$$

The quadratic approximation is:

$$f(0,0) + (xf_x + yf_y)\big|_{(0,0)} + \frac{1}{2}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})\big|_{(0,0)} = x^2+y^2.$$

Chap 14.10 Partial Derivatives with Constrained Values.

1. The procedure of finding $\frac{\partial w}{\partial x}$ when $w=f(x,y,z)$ are constrained by another equation.

(I) ① Decide the dependent and independent variables

② Eliminate the other dependent variable(s) in w

③ Differentiate as usual.

(II) ① The same in (I)

② Differentiate both the w and the constrained equation

③ Solve out the formula of $\frac{\partial w}{\partial x}$.

2. Notation.

$\left(\frac{\partial w}{\partial x}\right)_y$ $\frac{\partial w}{\partial x}$ with x, y independent

3. Arrow Diagrams.

Consider $w = x^2+y-z+\sin t$ and $x+y=t$, calculate $\left(\frac{\partial w}{\partial x}\right)_{y,z}$.

Solution:

$$\underset{\substack{\text{indpdt}\\\text{variables}}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} \longrightarrow \underset{\substack{\text{intermediate}\\\text{variables}}}{\begin{pmatrix} u \\ v \\ s \\ t \end{pmatrix}} \longrightarrow \underset{\substack{\text{dpdt}\\\text{variables}}}{w} \quad \text{where} \quad \begin{cases} u=x \\ v=y \\ s=z \\ t=x+y \end{cases}$$

$$w=u^2+v-s+\sin t$$

Then, $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial w}{\partial u} + \frac{\partial w}{\partial t} = 2x + \cos(x+y)$$

Exercise.

$(f \not\equiv 0)$

1. Prove: If $f(x,y,z)=0$, then $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$.

Proof: Calculate $\left(\frac{\partial x}{\partial y}\right)_z$ first:

take y, z as indpdt variables, x as dpdt variables, differentiate $f(x,y,z)=0$.

Then, $\frac{\partial}{\partial y}(f(x,y,z)) = f_1 \left(\frac{\partial x}{\partial y}\right)_z + f_2 = 0$

That is, $\left(\frac{\partial x}{\partial y}\right)_z = -\frac{f_2}{f_1}$, if $f_1 \neq 0$. (If $f_1=0$, then $f_2=0$. Then, $f_3=0$. Thus, $f \equiv 0$. Contradiction!)

Similarly, $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{f_3}{f_2}$; $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{f_1}{f_3}$

Then, $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$. □

2. Prove. If $z = x + f(u)$, where $u = xy$, show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = x$$

Proof: $\frac{\partial z}{\partial x} = 1 + f' \cdot y \qquad \frac{\partial z}{\partial y} = f' \cdot x$

$$x \cdot \frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x. \ \square$$

Additional Exercises:

1. Find the maximum value of $f(x,y)=6xye^{-(2x+3y)}$ in the closed first quadrant.

Solution: Since $\lim_{x\to+\infty} f(x,y) = \lim_{y\to+\infty} f(x,y) = 0$, $f(x,0)=f(0,y)=0$ and $f(x,y)>0$ in $\begin{cases} x>0 \\ y>0 \end{cases}$

then the maximum point of $f(x,y)$ will only be one of the local maximums.

$$\begin{cases} f_x = 6ye^{-(2x+3y)} - 12xye^{-(2x+3y)} = 0 \\ f_y = 6xe^{-(2x+3y)} - 18xye^{-(2x+3y)} = 0. \end{cases} \quad (x,y>0)$$

Then, $\begin{cases} x=\frac{1}{2} \\ y=\frac{1}{3} \end{cases}$.

Now we have only one local extreme $(\frac{1}{2},\frac{1}{3})$ in $\begin{cases} x>0 \\ y>0 \end{cases}$ and $f(\frac{1}{2},\frac{1}{3})>0$.

If $(\frac{1}{2},\frac{1}{3})$ is not the local maximum, then we will have some other local extreme, otherwise, there exists a sequence $\{(x_n,y_n)\}$ $(x_n\to\infty, y_n\to\infty)$ s.t. $\lim_{n\to\infty} f(x_n,y_n)>0$.

Contradiction!

Hence, $(\frac{1}{2},\frac{1}{3})$ is the only local maximum.

That is, $f(\frac{1}{2},\frac{1}{3})$ is the maximum of $f(x,y)$ in the closed first quadrant. $\square$

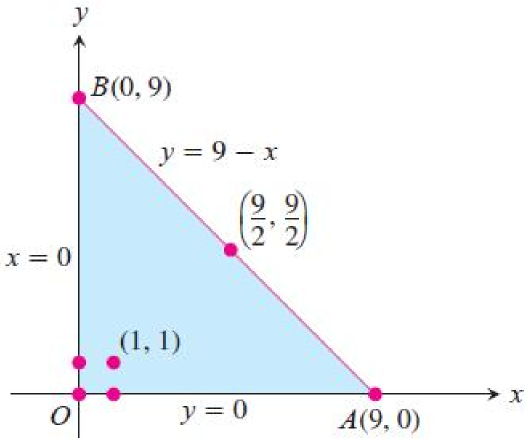

FIGURE 14.44 This triangular region is the domain of the function in Example 5.

EXAMPLE 5 Finding Absolute Extrema

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0$, $y = 9 - x$.

Solution Since f is differentiable, the only places where f can assume these values are points inside the triangle (Figure 14.44) where $f_x = f_y = 0$ and points on the boundary.

(a) Interior points. For these we have

$$f_x = 2 - 2x = 0, \qquad f_y = 2 - 2y = 0,$$

yielding the single point $(x, y) = (1, 1)$. The value of f there is

$$f(1, 1) = 4.$$

(b) Boundary points. We take the triangle one side at a time:

(i) On the segment OA, $y = 0$. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of x defined on the closed interval $0 \le x \le 9$. Its extreme values (we know from Chapter 4) may occur at the endpoints

$$x = 0 \quad \text{where} \quad f(0, 0) = 2$$
$$x = 9 \quad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61$$

and at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$

(ii) On the segment OB, $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 2y - y^2.$$

We know from the symmetry of f in x and y and from the analysis we just carried out that the candidates on this segment are

$$f(0, 0) = 2, \qquad f(0, 9) = -61, \qquad f(0, 1) = 3.$$

(iii) We have already accounted for the values of f at the endpoints of AB, so we need only look at the interior points of AB. With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2 = -61 + 18x - 2x^2.$$

Setting $f'(x, 9 - x) = 18 - 4x = 0$ gives

$$x = \frac{18}{4} = \frac{9}{2}.$$

At this value of x,

$$y = 9 - \frac{9}{2} = \frac{9}{2} \quad \text{and} \quad f(x, y) = f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}.$$

Summary We list all the candidates: $4, 2, -61, 3, -(41/2)$. The maximum is 4, which f assumes at $(1, 1)$. The minimum is -61, which f assumes at $(0, 9)$ and $(9, 0)$. ■

Solving extreme value problems with algebraic constraints on the variables usually requires the method of Lagrange multipliers in the next section. But sometimes we can solve such problems directly, as in the next example.